International Journal of Theoretical Physics, Vol. 42, No. 11, November 2003 (© 2003)

Quantum Set Theory

Satoko Titani[1,3] and Haruhiko Kozawa[2]

Received May 15, 2003

The complete orthomodular lattice of closed subspaces of a Hilbert space is considered as the logic describing a quantum physical system, and called a *quantum logic*. G. Takeuti developed a quantum set theory based on the quantum logic. He showed that the real numbers defined in the quantum set theory represent observables in quantum physics. We formulate the quantum set theory by introducing a strong implication corresponding to the lattice order, and represent the basic concepts of quantum physics such as propositions, symmetries, and states in the quantum set theory.

KEY WORDS: quantum logic; set theory; lattice-valued universe.

1. INTRODUCTION

The formulation of quantum physics in terms of lattice theory was first introduced by Birkhoff and von Neumann (1936). In the setting, a system of quantum physics is represented as a Hilbert space whose elements correspond to physical states while propoistions in quantum physics correspond to closed subspaces of the Hilbert space. A proposition of quantum physics is considered as a closed subspace of a Hilbert space consisting of states in which the proposition is certainly true. Thus, complete orthomodular lattice of closed subspaces of a Hilbert space may be considered as the logic describing a quantum physical system.

Let H be a Hilbert space consisting of physical states, and $P(H)$ be a complete orthomodular lattice consisting of all closed linear subspaces of H, or equivalently, consisting of all projections on H. $P(H)$ is called a *quantum logic*. The set theory developed in the $P(H)$-valued universe $V^{P(H)}$ by using the quantum logic is called a *quantum set theory*.

$P(H)$-valued universe $V^{P(H)}$ is constructed by induction as follows:

$$V_\alpha^{P(H)} = \left\{u \mid \exists\beta < \alpha\exists\mathcal{D}u \subset V_\beta^{P(H)}(u : \mathcal{D}u \to P(H))\right\},$$

[1] Research Institute for Information Science, Department of Natural Science and Mathematics, Chubu University, Kasugai, Aich, Japan.
[2] Department of Applied Physics, Chubu University, Kasugai, Aich, Japan.
[3] To whom correspondence should be addressed at Department of Natural Science and Mathematics, Chubu University, Kasugai, Aich, 487-8501, Japan; e-mail: titani@isc.chubu.ac.jp.

0020-7748/03/1100-2575/0

$$V^{P(H)} = \bigcup_{\alpha \in On} V_\alpha^{P(H)}.$$

G. Takeuti developed in Takeuti (1981) a quantum set theory in $V^{P(H)}$ based on the quantum logic $P(H)$. He showed in Takeuti (1978) that real numbers defined in the language of the quantum set theory represent self-adjoint operators acting on H. That is, real numbers in $V^{P(H)}$ (seen from outside $V^{P(H)}$) represent observables in quantum theory.

We use the notation $\underset{q}{\rightarrow}, \underset{q}{=}, \underset{q}{\in}$ instead of $\rightarrow, =, \in$ in Takeuti (1981), where

$$\varphi \rightarrow_q \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^\perp \vee (\varphi \wedge \psi).$$

The equality $\underset{q}{=}$ and membership relation $\underset{q}{\in}$ corresponding to the quantum implication $\rightarrow_q$ are interpreted in $V^{P(H)}$ by

$$[\![u \underset{q}{=} v]\!] = \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_q [\![x \underset{q}{\in} v]\!]) \wedge \bigwedge_{x \in \mathcal{D}u} (v(x) \rightarrow_q [\![x \underset{q}{\in} u]\!]),$$

$$[\![x \underset{q}{\in} v]\!] = \bigvee_{x \in \mathcal{D}v} [\![u \underset{q}{=} x]\!] \wedge v(x),$$

where $[\![A]\!]$ is the truth value of sentence A in $V^{P(H)}$.

Operator $\rightarrow_q$ is an implication in the sense that

$$[\![\varphi \wedge (\varphi \rightarrow_q \psi)]\!] \leq [\![\psi]\!].$$

But, unfortunately,

$$[\![\varphi \wedge \psi]\!] \leq [\![\xi]\!] \quad \text{does not imply} \quad [\![\varphi]\!] \leq [\![\psi \rightarrow_q \xi]\!],$$

unless $[\![\varphi]\!]$ and $[\![\psi]\!]$ are compatible, because of nondistributivity of lattice $P(H)$. It follows that the transitivity of $\underset{q}{=}$:

$$(u \underset{q}{=} v) \wedge (v \underset{q}{=} w) \rightarrow_q (u \underset{q}{=} w)$$

is too restrictive to develop a set theory. That is, equality axioms for $\underset{q}{=}$ are not valid in $V^{P(H)}$.

In order to restore the equality axioms, we introduce stronger implication $\rightarrow$ called *basic implication*, which represents the lattice order:

$$(a \rightarrow b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise,} \end{cases}$$

where 1, 0 are the top and bottom of the complete lattice.

In Titani (1999), we formulated a lattice-valued logic and a lattice-valued set theory, by introducing the basic implication $\rightarrow$. The lattice-valued logic is the logical counterpart of complete lattice. The completeness of the lattice-valued logic was proved in Takano (2002). For an arbitrary complete lattice $\mathcal{L}$, the $\mathcal{L}$-valued universe $V^{\mathcal{L}}$ is a model of lattice-valued set theory based on the lattice-valued logic. Since $P(H)$ is a complete lattice, $V^{P(H)}$ is a model of the lattice-valued set theory. The quantum set theory on $V^{P(H)}$ is formulated as a lattice-valued set theory with the quantum implication $\rightarrow_q$ as well as the basic implication $\rightarrow$.

Remark 1. The basic implication is not the only interpretation of implication for lattice-valued logic. For example, let $\mathcal{L}$ be an orthomodular complete lattice and Z be the center of $\mathcal{L}$, that is, Z be the complete Boolean subalgebra which consists of all $a \in \mathcal{L}$ distributive over all subsets of $\mathcal{L}$.

$$Z = \left\{ a \in \mathcal{L} \mid \forall \{b_\alpha\} \subset \mathcal{L} \left(a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha (a \wedge b_\alpha),\, a \vee \bigwedge_\alpha b_\alpha = \bigwedge_\alpha (a \vee b_\alpha) \right) \right\}$$

Then $\mathcal{L}$ is also a model of lattice-valued logic with the following interpretation of implication on $\mathcal{L}$:

$$(a \rightarrow b) = \bigvee \{z \in Z \mid a \wedge z \leq b\}.$$

Elements of the Boolean-valued subuniverse $V^{\mathbf{2}} \subset V^{P(H)}$ are called check sets. The set of rational numbers defined in the universe $V^{P(H)}$ is a check set $\check{\mathbb{Q}}$ corresponding to the set $\mathbb{Q}$ of rational numbers. A *real number* is defined in $V^{P(H)}$ as an upper part u of a Dedekind cut of rational numbers $\check{\mathbb{Q}}$. Complex numbers are defined in $V^{P(H)}$ as pairs of compatible real numbers. An element u of $V^{P(H)}$ such that "u is a real (complex) number" holds in $V^{P(H)}$, is called a *quantum real (complex) number*.

Each proposition, which is considered as a projection on H, is represented as a quantum real number u such that $[\![u \underset{q}{=} \check{1} \vee u \underset{q}{=} \check{0}]\!] = 1$. A symmetry, which is considered as a unitary operator on H, is represented as a quantum complex number. A state of the physical system, which is considered as a unit vector in H, is represented as a set of quantum complex numbers indexed by unitary operators on H.

In Section 2, 3, we review proposition system in Piron (1976) and lattice-valued set theory in Titani (1999). A quantum set theory is formulated in Section 4. Real and complex numbers in $V^{P(H)}$, that is, quantum real and complex numbers, are discussed in Section 5. Then, in Section 6, physical concepts such as propositions, symmetries, and states are represented in the universe $V^{P(H)}$.

2. PRELIMINARIES

In this section, we review a formulation of quantum physics by using the language of lattice, in Piron (1976).

2.1. Proposition System

Complete lattice $\mathcal{L}$ satisfying the following axoims (C), (P), (A) is called a *proposition system*, and elements of $\mathcal{L}$ are called *propositions*. The top and bottom elements of complete lattice $\mathcal{L}$ will be denoted by 1 and 0, respectively.

$$\bigvee \mathcal{L} = 1, \qquad \bigvee \emptyset = 0$$

(C): For each $c \in \mathcal{L}$ there exists a unique orthocomplement $c^\perp \in \mathcal{L}$ such that
- (C_1) $c^{\perp\perp} = c$
- (C_2) $c \vee c^\perp = 1$, and $c \wedge c^\perp = 0$
- (C_3) $b \leq c \Longrightarrow c^\perp \leq b^\perp$ for $\forall b \in \mathcal{L}$

(P): If $b, c \in \mathcal{L}$ and $b \leq c$, then the sublattice of $\mathcal{L}$ generated by $\{b, b^\perp, c, c^\perp\}$ is a distributive lattice.

A complete lattice satisfying (C) and (P) is called a *complete orthomodular lattice*.

If $b \neq c$ and $b \leq c$, one say that *c covers b* when

$$b \leq x \leq c \implies x = b \quad \text{or} \quad x = c.$$

A proposition which covers 0 is called an *atom*.

(A): (A_1) If b is a proposition different from 0, there exists an atom $p \leq b$.
(A_2) If p is an atom and if $p \wedge b = 0$, then $p \vee b$ covers b.

A system of quantum physics is represented as a proposition system. Let $\mathcal{L}$ be the proposition system. An observable is defined as a correspondence between the propositions associated with the measuring apparatus and propositions associated with the physical system. Thus, a c-morphism of a complete Boolean algebra into the proposition system is called an *observable,* where *c-morphism* is a mapping σ of a complete orthocomplemented lattice $\mathcal{L}_1$ into a complete orthocomplemented lattice $\mathcal{L}_2$ such that

1. $\sigma(\bigvee_i b_i) = \bigvee_i (\sigma b_i)$,
2. $b \perp c \Longrightarrow \sigma b \perp \sigma c$.

2.2. Compatibility

Definition 2.1. Elements b, c of a complete orthomodular lattice $\mathcal{L}$ are said to be *compatible*, in symbols $c \natural b$, if the sublattice generated by $\{b, b^\perp, c, c^\perp\}$ is distributive. For $b \in \mathcal{L}$ and a subset A of $\mathcal{L}$,

$$b \natural A \overset{\text{def}}{\Longleftrightarrow} \forall a \in A (b \natural a).$$

Theorem 2.1. *(Piron (1976) pp. 25–27). For elements b,c of a complete orthomodular lattice $\mathcal{L}$, the following conditions are equivalent.*

1. *b,c are compatible*
2. $(b \wedge c) \vee (b^{\perp} \wedge c) \vee (b \wedge c^{\perp}) \vee (b^{\perp} \wedge c^{\perp}) = 1$
3. $(b \wedge c) \vee (b^{\perp} \wedge c) = c$
4. $(b \vee c^{\perp}) \wedge c = b \wedge c$

Theorem 2.2. *(Piron (1976) p. 27). For a subset C of a complete orthomodular lattice $\mathcal{L}$ and $b \in \mathcal{L}$, if $b \,\text{\textdaggerdbl}\, C$ then*

$$\bigvee_{c \in C} (b \wedge c) = b \wedge \left(\bigvee C\right), \qquad \bigwedge_{c \in C} (b \vee c) = b \vee \left(\bigwedge C\right).$$

Theorem 2.3. *(Piron (1976) p. 28). For a subset C of a complete orthomodular lattice $\mathcal{L}$ and $b \in \mathcal{L}$, if $b \,\text{\textdaggerdbl}\, C$ then*

$$b \,\text{\textdaggerdbl} \bigvee C \quad \textit{and} \quad b \,\text{\textdaggerdbl} \bigwedge C.$$

Definition 2.2. Let $\mathcal{L}$ be a complete orthomodular lattice. For $a, b \in \mathcal{L}$,

$$(a \rightarrow_q b) \stackrel{\text{def}}{=} a^{\perp} \vee (a \wedge b).$$

Then we have

Theorem 2.4. *(Takeuti (1981) p. 305). If $a, b, c \in \mathcal{L}$ and $a \,\text{\textdaggerdbl}\, c$, then*

$$c \leq (a \rightarrow_q b) \iff a \wedge c \leq b.$$

2.3. Direct Union

Definition 2.3. Direct union $\bigvee_{\alpha} \mathcal{L}_{\alpha}$ of a family $\{\mathcal{L}_{\alpha}\}$ of proposition systems is the complete orthocomplemented lattice consisting of families $\{x_{\alpha}\}$ where $x_{\alpha} \in \mathcal{L}_{\alpha}$ with the ordering defined by

$$\{x_{\alpha}\} \leq \{y_{\alpha}\} \stackrel{\text{def}}{\iff} \forall_{\alpha}(x_{\alpha} \leq y_{\alpha})$$

and orthocomplementation defined by

$$\{x_{\alpha}\}^{\perp} \stackrel{\text{def}}{\iff} \{x_{\alpha}^{\perp}\}.$$

Theorem 2.5. *(Piron (1976) p. 34). In the direct union $\bigvee_{\alpha} \mathcal{L}_{\alpha}$ of a family $\{\mathcal{L}_{\alpha}\}$ of propositon systems,*

$$\{x_{\alpha}\} \,\text{\textdaggerdbl}\, \{y_{\alpha}\} \quad \textit{if and only if} \quad \forall_{\alpha}(x_{\alpha} \,\text{\textdaggerdbl}\, y_{\alpha})$$

Definition 2.4. A proposition system $\mathcal{L}$ is said to be *irreducible* if its center consists of 0 and 1, i.e.

$$\mathcal{L} \text{ is irreducible } \overset{\text{def}}{\Longleftrightarrow} Z = \{0, 1\},$$

where the center Z of $\mathcal{L}$ is the set of elements compatible with all elements of $\mathcal{L}$.

$$Z = \{ z \in \mathcal{L} \mid \forall_a \in \mathcal{L}(z \rfloor a)\}.$$

Theorem 2.6. *(Piron (1976) p. 35). Every proposition system $\mathcal{L}$ is the direct union of irreducible proposition systems.*

2.4. Standard Proposition System $P(H)$

A system of quantum physics is represented as a Hilbert space whose elements correspond to physical states. Propositions of the quantum physics are represented as closed subspaces of the Hilbert space, which forms a proposition system. Proposition system $P(H)$ consisting of closed subspaces of a separable Hilbert space H, or equivalently consisting of all projections on H is called a *standard proposition system*. In what follows we deal with a standard proposition system $P(H)$ unless otherwise mentioned.

3. LATTICE-VALUED SET THEORY

Proposition system $P(H)$ is a complete lattice. Thus, a lattice-valued set theory based on lattice-valued logic is developed in $P(H)$-valued universe $V^{P(H)}$ (Titani, 1999). In Section 3.1, the lattice-valued set theory is reviewed briefly.

3.1. Lattice-Valued Universe

Let $\mathcal{L}$ be any complete lattice. $\mathcal{L}$-valued universe $V^{\mathcal{L}}$ of lattice-valued set theory is constructed by induction:

$$V_\alpha^{\mathcal{L}} = \{u \mid \exists\beta < \alpha \exists \mathcal{D}u \subset V_\beta^{\mathcal{L}}(u : \mathcal{D}u \to \mathcal{L})\},$$

$$V^{\mathcal{L}} = \bigcup_{\alpha \in On} V_\alpha^{\mathcal{L}}.$$

The least α such that $u \in V_\alpha^{\mathcal{L}}$ is called the *rank* of u. On the complete lattice $\mathcal{L}$, operations $\to$ and $\neg$ are defined by

$$(a \to b) \overset{\text{def}}{=} \bigvee\{ x \in \mathbf{2} \mid a \wedge x \leq b \} = \begin{cases} 1, & \text{if } a \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\neg a \overset{\text{def}}{=} (a \to 0) = \bigvee \{ x \in \mathbf{2} \mid a \wedge x \leq 0 \} = \begin{cases} 1, & \text{if } a = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Formulas of lattice-valued set theory are constructed from atomic formula of the form $u = v$ or $u \in v$ by logical operators $\wedge, \vee, \neg, \to, \forall, \exists$. Atomic formulas are interpreted in $V^{\mathcal{L}}$ as

$$[\![u = v]\!] = \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to [\![x \in u]\!]),$$

$$[\![u \in v]\!] = \bigvee_{x \in \mathcal{D}v} (v(x) \wedge [\![u = x]\!]).$$

The logical operators are interpreted as the corresponding lattice opertors on $\mathcal{L}$. The following abbreviations will be used.

$$\Box\varphi \overset{\text{def}}{\iff} ((\varphi \to \varphi) \to \varphi).$$

Then

$$[\![\Box\varphi]\!] = \begin{cases} 1 & \text{if } [\![\varphi]\!] = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We say an element p of $\mathcal{L}$ is $\Box$-*closed if* $p = \Box p$.

Lemma 3.1. (Titani (1999)). *For every* $u, v \in V^{\mathcal{L}}$,

1. $[\![u = v]\!]$ *is* $\Box$-*closed,*
2. $[\![u = v]\!]$ *is distributive over any subset* $\{b_k\}_k$ *of* $\mathcal{L}$:
   $(\bigvee_k b_k) \wedge [\![u = v]\!] = \bigvee_k (b_k \wedge [\![u = v]\!])$.

Lemma 3.2. (Titani (1999)). *For every* $u, v, w \in V^{\mathcal{L}}$,

1. $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$
2. $[\![u \in w]\!] \wedge [\![u = v]\!] \leq [\![v \in w]\!]$
3. $[\![w \in u]\!] \wedge [\![u = v]\!] \leq [\![w \in v]\!]$

Definition 3.1.

$$\forall x{\in}u\varphi(x) \overset{\text{def}}{\iff} \forall x(x \in u \to \varphi(x)), \qquad \exists x{\in}u\varphi(x) \overset{\text{def}}{\iff} \exists x(x \in u \wedge \varphi(x)).$$

Lemma 3.3. (Titani (1999)). *For a formula* $\varphi(a)$,

$$[\![\forall x{\in}u\varphi(x)]\!] = \bigwedge_{x \in \mathcal{D}u} [\![x \in u \to \varphi(x)]\!], \quad [\![\exists x{\in}u\varphi(x)]\!] = \bigvee_{x \in \mathcal{D}u} [\![x{\in}u \wedge \varphi(x)]\!].$$

Theorem 3.1. (Titani (1999)). *The following A1–A11 are valid in $V^{\mathcal{L}}$.*

A1. ***Equality:*** $\forall u \forall v(u = v \wedge \varphi(u) \to \varphi(v))$.

A2. ***Extensionality:*** $\forall u, v(\forall x(x \in u \leftrightarrow x \in v) \to u = v)$.

A3. ***Pairing:*** $\forall u, v \exists z(\forall x(x \in z \leftrightarrow (x = u \vee x = v)))$.

The set z satisfying $\forall x(x \in z \leftrightarrow (x = u \vee x = v))$ *is denoted by* $\{u, v\}$.

A4. ***Union:*** $\forall u \exists z(\forall x(x \in z \leftrightarrow \exists y \in u(x \in y)))$.

The set z satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ *is denoted by* $\bigcup u$.

A5. ***Power set:*** $\forall u \exists z(\forall x(x \in z \leftrightarrow x \subset u))$ *where* $x \subset u \overset{def}{\iff} \forall y(y \in x \to y \in u)$. *The set z satisfying* $\forall x(x \in z \leftrightarrow x \subset u)$ *is denoted by* $\mathcal{P}(u)$.

A6. ***Infinity:*** $\forall u(\exists x(x \in u) \wedge \forall x \in u \exists y \in u(x \in y))$.

A7. ***Separation:*** $\forall u \exists v(\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x)))$.
The set v satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ *is denoted by* $\{x \in u \mid \varphi(x)\}$.

A8. ***Collection:*** $\forall u \exists v(\forall x \in u \exists y \varphi(x, y) \to \forall x \in u \exists y \Box(y \in v \wedge \varphi(x, y))$.

A9. ***$\in$-induction:*** $\forall x(\forall y \in x \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$.

A10. ***Zorn:*** $\forall x \in u \Box(x \in u) \wedge \forall u(Chain(v, u) \to \bigcup v \in u) \to \exists z\, Max(z, u)$, *where*

$$Chain(v, u) \overset{def}{\iff} v \subset u \wedge \forall x, y \in v(x \subset y \vee y \subset x),$$

$$Max(z, u) \overset{def}{\iff} z \in u \wedge \forall x \in u(z \subset x \to z = x).$$

A11. ***Axiom of $\Diamond$:*** $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u))$,*where* $\Diamond\varphi \overset{def}{\iff} (\varphi \to \bot) \to \bot$
The set z satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ *is denoted by* $\Diamond u$.

Definition 3.2. $u \in V^{\mathcal{L}}$ is said to be *definite* if $u(x) = [\![x \in u]\!]$ for all $x \in \mathcal{D}u$.

Lemma 3.4. *For any* $u \in V^{\mathcal{L}}$ *there exists a definte* $v \in V^{\mathcal{L}}$ *such that* $[\![u = v]\!] = 1$.

Proof: For $u \in V^{\mathcal{L}}$, let

$$\mathcal{D}v = \mathcal{D}u, \quad v(x) = [\![x \in u]\!].$$

Then v is definte and $[\![u = v]\!] = 1$. □

In what follows we may assume that each $u \in V^{\mathcal{L}}$ is definite.

3.2. Embedding of Lattice-Valued Universes

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices. We denote the top $\bigvee \mathcal{L}_i$ of $\mathcal{L}_i$ by $1_{\mathcal{L}_i}(i = 1, 2)$. An embedding σ of $\mathcal{L}_1$ into $\mathcal{L}_2$ is said to be *unital* if $\sigma(1_{\mathcal{L}_1}) = 1_{\mathcal{L}_2}$, and *sup-preserving* if $\sigma(\bigvee A) = \bigvee_{a\in A} \sigma(a)$ for $A \subset \mathcal{L}_1$.

If $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$ is a unital sup-preserving embedding, then

$$a \le b \iff \sigma(a) \le \sigma(b) \quad \text{and} \quad \sigma(a \to b) = (\sigma(a) \to \sigma(b)) \quad \text{for} \quad a, b \in \mathcal{L}_1.$$

Definition 3.3. A unital sup-preserving embedding $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$ is extended to embedding $\sigma : V^{\mathcal{L}_1} \to V^{\mathcal{L}_2}$ defined by

$$\begin{cases} \mathcal{D}(\sigma u) & = \{\sigma x \mid x \in \mathcal{D}u\} \\ (\sigma u)(\sigma x) & = \bigvee \{\sigma [\![x' \in u]\!]_{\mathcal{L}_1} \mid x' \in \mathcal{D}u, \quad [\![\sigma x = \sigma x']\!]_{\mathcal{L}_2} = 1\} \end{cases} \tag{3.1}$$

where $[\![\quad]\!]_{\mathcal{L}_i}$ is the truth value in $V^{\mathcal{L}_i}$ for $i = 1, 2$. The mapping $\sigma : V^{\mathcal{L}_1} \to V^{\mathcal{L}_2}$ is called the *embedding associated with* $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$.

Theorem 3.2. *Let $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$ be a unital sup-preserving embedding, and $\sigma : V^{\mathcal{L}_1} \to V^{\mathcal{L}_2}$ be the embedding associated with $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$. Then for $u, v \in V^{\mathcal{L}_1}$,*

$$\sigma [\![u = v]\!]_{\mathcal{L}_1} = [\![\sigma u = \sigma v]\!]_{\mathcal{L}_2} \quad \textit{and} \quad \sigma [\![u \in v]\!]_{\mathcal{L}_1} = [\![\sigma u \in \sigma v]\!]_{\mathcal{L}_2}.$$

Proof: We assume the theorem for $x, y \in V^{\mathcal{L}_1}_{<\alpha}$. For the first part, it suffices to show that

$$[\![\sigma u = \sigma v]\!]_{\mathcal{L}_2} = 1 \iff [\![u = v]\!]_{\mathcal{L}_1} = 1.$$

By the induction hypothesis,

$$[\![\sigma x = \sigma x']\!]_{\mathcal{L}_2} = 1 \iff [\![x = x']\!]_{\mathcal{L}_1} = 1 \quad \text{for} \quad x, x' \in \mathcal{D}u.$$

It follows that $(\sigma u)(\sigma x) = \sigma [\![x \in u]\!]_{\mathcal{L}_1}$ for $x \in \mathcal{D}u$. If $[\![u = v]\!] = 1$, then for $x \in \mathcal{D}u$,

$$\begin{aligned} (\sigma u)(\sigma x) &= \sigma [\![x \in u]\!]_{\mathcal{L}_1} \le \sigma [\![x \in v]\!]_{\mathcal{L}_1} \\ &= \sigma \left(\bigvee_{y \in \mathcal{D}v} [\![x = y]\!] \wedge v(y) \right) \le \bigvee_{y \in \mathcal{D}v} [\![\sigma x = \sigma y]\!]_{\mathcal{L}_2} \wedge (\sigma v)(\sigma y) \\ &\le [\![\sigma x \in \sigma v]\!]_{\mathcal{L}_2} \quad \text{for} \quad x \in \mathcal{D}u. \end{aligned}$$

Symmetrically, $(\sigma v)(\sigma x) \le [\![\sigma x \in \sigma u]\!]_{\mathcal{L}_2}$ for $x \in \mathcal{D}v$. Therefore, $[\![\sigma u = \sigma v]\!]_{\mathcal{L}_2} = 1$. Conversely, if $[\![\sigma u = \sigma v]\!]_{\mathcal{L}_2} = 1$, then for $x \in \mathcal{D}u$,

$$\begin{aligned} & \sigma [\![x \in u]\!]_{\mathcal{L}_1} = (\sigma u)(\sigma x) \le [\![\sigma x \in \sigma v]\!]_{\mathcal{L}_2} = \sigma [\![x \in v]\!]_{\mathcal{L}_1}, \\ & \therefore [\![x \in u]\!]_{\mathcal{L}_1} \le [\![x \in v]\!]_{\mathcal{L}_1}. \end{aligned}$$

Symmetrically, $[\![x \in v]\!]_{\mathcal{L}_1} \leq [\![x \in u]\!]_{\mathcal{L}_1}$ for $x \in \mathcal{D}v$. Therefore, $[\![u = v]\!]_{\mathcal{L}_1} = 1$. Thus,

$$\sigma[\![u = v]\!]_{\mathcal{L}_1} = [\![\sigma u = \sigma v]\!]_{\mathcal{L}_2} \tag{3.2}$$

By using (3.2) we have

$$\begin{aligned}\sigma[\![u \in v]\!]_{\mathcal{L}_1} &= \bigvee_{x \in \mathcal{D}v} (\sigma[\![u = x]\!]_{\mathcal{L}_1} \wedge \sigma[\![x \in v]\!]_{\mathcal{L}_1}) \\ &= \bigvee_{x \in \mathcal{D}v} ([\![\sigma u = \sigma x]\!]_{\mathcal{L}_2} \wedge (\sigma u)(\sigma x)) = [\![\sigma u \in \sigma v]\!]_{\mathcal{L}_2}.\end{aligned}$$

□

Corollary 3.1. *Let $\sigma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ be a unital sup-preserving embedding such that $\sigma(p \wedge q) = \sigma p \wedge \sigma p$ for $p, q \in \mathcal{L}_1$, and $\sigma : V^{\mathcal{L}_1} \rightarrow V^{\mathcal{L}_2}$ be the embedding associated with $\sigma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$. If $\varphi(x_1, x_2, \ldots, x_n)$ is a bounded formula of lattice-valued set theory, and $u_1, u_2, \ldots, u_n \in V^{\mathcal{L}_1}$, then*

$$\sigma[\![\varphi(u_1, u_2, \ldots, u_n)]\!]_{\mathcal{L}_1} = [\![\varphi(\sigma u_1, u_2, \ldots, \sigma u_n)]\!]_{\mathcal{L}_2}.$$

Proof: By induction on the complexity of φ. Since other cases are obvious, we prove only the case that $\varphi(u_1, u_2, \ldots, u_n)$ is of the form $\forall x \in u \varphi(x, u_1, u_2, \ldots, u_n)$. If $\{p_i\}$ is a set of □-closed formulas, then $\sigma(\bigwedge_i p_i) = \bigwedge_i \sigma(p_i)$. Hence,

$$\begin{aligned}\sigma[\![\forall x \in u \varphi(x, u_1, u_2, \cdots, u_n)]\!] &= \sigma\left(\bigwedge_{x \in \mathcal{D}u} [\![x \in u \rightarrow \varphi(x, u_1, u_2, \cdots, u_n)]\!]\right) \\ &= \bigwedge_{x \in \mathcal{D}u} \sigma[\![x \in u \rightarrow \varphi(x, u_1, u_2, \cdots, u_n)]\!]) \\ &= \bigwedge_{x \in \mathcal{D}u} [\![\sigma x \in \sigma u \rightarrow \varphi(\sigma x, \sigma u_1, \sigma u_2, \cdots, \sigma u_n)]\!]) \\ &= [\![\forall x \in \sigma u(x, \sigma u_1, \sigma u_2, \cdots, \sigma u_n)]\!]).\end{aligned}$$

□

3.3. Check Sets

Let V be a universe of ZFC in which lattice-valued universe $V^{\mathcal{L}}$ is constructed. The subset $\mathbf{2}(=\{1, 0\})$ of $\mathcal{L}$ is a Boolean subalgebra of $\mathcal{L}$, and the universe V is isomorphic to $V^{\mathbf{2}}$.

Definition 3.4. Elements of the subuniverse $V^{\mathbf{2}}$ of $V^{\mathcal{L}}$ are called *check sets*.

The check set corresponding to $u \in V$ is denoted by $\check{u}$. The check set $\check{u}$ is defined by

$$\begin{cases} \mathcal{D}\check{u} &= \{\check{x} \mid x \in u\} \\ \check{u}(\check{x}) = 1 \end{cases}$$

"x is a check set," in symbols $\mathrm{ck}(x)$, can be expressed in the language of lattice-valued set theory (Titani, 1999):

$$\mathrm{ck}(x) \iff \forall t[t \in x \leftrightarrow \Box(t \in x) \wedge \mathrm{ck}(t)].$$

Identity mapping $I : \mathbf{2} \to \mathcal{L}$ is a unital sup-preserving embedding, and V is embedded in $V^{\mathcal{L}}$, by

$$u \mapsto \check{u} \in V^2 \subset V^{\mathcal{L}} \quad (u \in V).$$

By Corollary 3.1 we have

Theorem 3.3. *If $\varphi(u_1, \ldots, u_n)$ is a bounded sentence of ZF set theory with constants $u_1, \ldots, u_n$ in V, then*

$$\varphi(u_1, \ldots, u_n) \iff [\![\varphi(\check{u}_1, \ldots, \check{u}_n)]\!] = 1.$$

3.4. $P(H)$-Valued Universe $V^{P(H)}$

The proposition system $P(H)$ is a complete lattice, and $P(H)$-valued universe $V^{P(H)}$ is constructed as a lattice-valued universe. G. Takeuti developed a quantum set theory in $V^{P(H)}$ by using unary operation $^{\perp}$ as negation, and implication $\to_q$ defined by

$$\varphi \to_q \psi \overset{\text{def}}{\iff} \varphi^{\perp} \vee (\varphi \wedge \psi).$$

$\to_q$ will be called *quantum implication* to distinguish from the basic implication $\to$. Equality $\underset{q}{=}$ and membership relation $\underset{q}{\in}$ corresponding to the quantum implication $\to_q$ are defined by

$$u \underset{q}{=} v \overset{\text{def}}{\iff} \forall x(x \in u \to_q x \underset{q}{\in} v) \wedge \forall x(x \in v \to_q x \underset{q}{\in} u)$$

$$u \underset{q}{\in} v \overset{\text{def}}{\iff} \exists(x \in v \wedge u \underset{q}{=} x).$$

By equaity axiom we have

Lemma 3.5. *For $u, v \in V^{P(H)}$, $[\![u = v]\!] \le [\![u \underset{q}{=} v]\!]$ and $[\![u \in v]\!] \le [\![u \underset{q}{\in} v]\!]$.*

4. A FORMULATION OF QUANTUM SET THEORY

In this paper *quantum logic* is formulated as a lattice-valued logic on $P(H)$ with new logical operator $^{\perp}$.

4.1. Language of Quantum Set Theory

Primitive symbols are:

1. Indivdual variables: $x, y, z, \ldots$
2. Predicate constants: $=, \in,$
3. Logical symbols: $\wedge, \vee, \rightarrow, \neg, \perp, \forall, \exists$.
4. Parentheses: (,)

Atomic formaulas are expressions of the form $t_1 = t_2$ or $t_1 \in t_2$ with terms t_1, t_2. *Formulas* are constructed from atomic formulas, by using the logical symbols. To denote formulas, we use

$$\varphi, \psi, \xi, \ldots, \varphi(x), \ldots.$$

Definition 4.1. $\Box$*-closed formulas* are defined inductively by

1. A formula of the form $\varphi \rightarrow \psi$ or $\neg\varphi$ is $\Box$-closed.
2. If formulas φ and ψ are $\Box$-closed, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$ and $\varphi^{\perp}$ are $\Box$-closed.
3. If a formulas $\varphi(x)$ is a $\Box$-closed formula with free variable x, then $\forall x\varphi(x)$ and $\exists x\varphi(x)$ are $\Box$-closed.
4. $\Box$-closed formulas are only those obtained by 1–3.

We use $\Gamma, \Delta, \Pi, \Lambda, \ldots$ to denote finite sequences of formulas; $\overline{\varphi}, \overline{\psi}, \ldots$ to denote $\Box$-closed formulas; and $\overline{\Gamma}, \overline{\Delta}, \overline{\Pi}, \overline{\Lambda}, \ldots$ to denote finite sequences of $\Box$-closed formulas.

A formal expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. The inference rules of the quantum logic are obtained from rules of lattice-valued logic in Titani (1999) by adding axioms for orthogonalization $^{\perp}$.

Axiom schema of lattice-valued logic: $\varphi \Rightarrow \varphi$

Structural rules:

Thinning: $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

Contraction: $\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

Interchange: $\dfrac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$

Cut: $\dfrac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \overline{\Delta}, \Lambda} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \overline{\Pi} \Rightarrow \Lambda}{\Gamma, \overline{\Pi} \Rightarrow \Delta, \Lambda}$

$$\dfrac{\Gamma \Rightarrow \Delta, \overline{\varphi} \quad \overline{\varphi}, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical rules:

$$\neg : \dfrac{\Gamma \Rightarrow \overline{\Delta}, \varphi}{\neg\varphi, \Gamma \Rightarrow \overline{\Delta}} \quad \dfrac{\Gamma \Rightarrow \Delta, \overline{\varphi}}{\neg\overline{\varphi}, \Gamma \Rightarrow \Delta} \qquad \dfrac{\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}}{\overline{\Gamma} \Rightarrow \overline{\Delta}, \neg\varphi} \quad \dfrac{\overline{\varphi}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\overline{\varphi}}$$

$$\wedge : \dfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \dfrac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \Gamma \Rightarrow \overline{\Delta}, \psi}{\Gamma \Rightarrow \overline{\Delta}, \varphi \wedge \psi}$$

$$\dfrac{\Gamma \Rightarrow \Delta, \overline{\varphi} \quad \Gamma \Rightarrow \Delta, \overline{\psi}}{\Gamma \Rightarrow \Delta, \overline{\varphi} \wedge \psi}$$

$$\vee : \dfrac{\varphi, \overline{\Gamma} \Rightarrow \Delta \quad \psi, \overline{\Gamma} \Rightarrow \Delta}{\varphi \vee \psi, \overline{\Gamma} \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \dfrac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\dfrac{\overline{\varphi}, \Gamma \Rightarrow \Delta \quad \overline{\psi}, \Gamma \Rightarrow \Delta}{\overline{\varphi} \vee \overline{\psi}, \Gamma \Rightarrow \Delta}$$

$$\rightarrow : \dfrac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \psi, \overline{\pi} \Rightarrow \Lambda}{(\varphi \rightarrow \psi), \Gamma, \overline{\pi} \Rightarrow \overline{\Delta}, \Lambda} \qquad \dfrac{\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \psi}{\overline{\Gamma} \Rightarrow \overline{\Delta}, (\varphi \rightarrow \psi)}$$

$$\forall : \dfrac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\varphi(x), \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \overline{\Delta}, \varphi(a)}{\Gamma \Rightarrow \overline{\Delta}, \forall x\varphi(x)} \quad \dfrac{\Gamma \Rightarrow \Delta, \overline{\varphi}(a)}{\Gamma \Rightarrow \Delta, \forall x\overline{\varphi}(x)}$$

where t is any term where a is a free variable which does not occur in the lower sequent.

$$\exists : \dfrac{\varphi(a), \overline{\Gamma} \Rightarrow \Delta}{\exists x\varphi(x), \overline{\Gamma} \Rightarrow \Delta} \quad \dfrac{\overline{\varphi}(a), \Gamma \Rightarrow \Delta}{\exists x\overline{\varphi}(x), \Gamma \Rightarrow \Delta} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\varphi(x)}$$

where a is a free variable which does not occur in the lower sequent. where t is any term

The above-mentioned are inference rules of lattice-valued set theory. For quantum logic, we postulate the following (C) and (P) corresponding orthomodularity of proposition system.

C: $(C1)$ $\varphi \iff \varphi^{\perp\perp}$
$(C2)$ $\Rightarrow \varphi \vee \varphi^{\perp},\ \varphi \wedge \varphi^{\perp} \Rightarrow$
$(C3)$ $(\varphi \rightarrow \psi) \Rightarrow \psi^{\perp} \rightarrow \psi^{\perp}$

P: $(\varphi \to \psi), \psi \Rightarrow \psi \wedge \varphi, \psi \wedge \varphi^{\perp}$.

The property (A) of proposition system is postulated as nonlogical axiom A12 in Section 6.1.

Definition 4.2. Formulas φ and ψ are said to be *compatible*, in symbols $\varphi \downarrow \psi$, if

$$\varphi \to (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}).$$

4.2. Nonlogical Axioms of Quantum Set Theory

The axioms A1–A11 of lattice-valued set theory in Theorem 3.1 are valid in $V^{P(H)}$. We adopt those A1–A11 and additional axiom A12 as nonlogical axioms of quantum set theory. Thus, theorems of lattice-valued set theory are valid in $V^{P(H)}$.

5. REAL AND COMPLEX NUMBERS IN $V^{P(H)}$

The set $\mathbb{N}$ of all natural numbers is constructed from the empty set by the successor function $x \mapsto x \cup \{x\}$. Integers are constructed as equivalence classes of pairs of natural numbers, and rational numbers as equivalence classes of pairs of integers. The real numbers are defined as upper parts of Dedekind cuts of rational numbers. Complex numbers are defined as pairs of real numbers. We denote the set of all integers, rational numbers, real numbers, and complex numbers in the universe V of classical set theory ZFC, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively.

5.1. Definition of Quantum Real Numbers

It is known that the sets of all natural numbers, integers, and rational numbers defined in a lattice-valued set theory coincide with check sets $\check{\mathbb{N}}$, $\check{\mathbb{Z}}$, $\check{\mathbb{Q}}$ corresponding to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, respectively. Upper part of Dedekind cut is a subsets of $\check{\mathbb{Q}}$, which is not necessarily a check set. "u is a subset of $\check{\mathbb{Q}}$," in symbols $u \subset \check{\mathbb{Q}}$, is defined by

$$u \subset \check{\mathbb{Q}} \overset{\text{def}}{\Longleftrightarrow} \forall x(x \in u \to x \in \check{\mathbb{Q}}).$$

Lemma 5.1. *If $[\![u \subset \check{\mathbb{Q}}]\!] = 1$ in $V^{P(H)}$, then there exists v in $V^{P(H)}$ such that*

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\} = \mathcal{D}\check{\mathbb{Q}} \quad \textit{and} \quad [\![u = v]\!] = 1.$$

Proof: Note that $[\![x = y]\!] \in \mathbf{2}$ and $[\![x = y]\!] \wedge \bigvee_i a_i = \bigvee_i([\![x = y]\!] \wedge a_i)$ for $\forall\{a_i\} \subset Q$. Define v by

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\}, \quad v(\check{r}) = [\![\check{r} \in u]\!] \quad \text{for} \quad r \in \mathbb{Q}.$$

If $x \in \mathcal{D}u$,

$$u(x) \leqslant [\![x \in \check{Q}]\!] \wedge [\![x \in u]\!] = \bigvee_{r \in \mathbb{Q}} ([\![x = \check{r}]\!] \wedge [\![x \in u]\!])$$

$$= \bigvee_{r \in \mathbb{Q}} ([\![x = \check{r}]\!] \wedge [\![\check{r} \in u]\!]) \leq [\![x \in v]\!]$$

Therefore, $[\![v = u \subset \check{\mathbb{Q}}]\!] = 1$. □

In what follows, "u is a subset of $\check{\mathbb{Q}}$", in symbols $u \subset \check{\mathbb{Q}}$ means that $\mathcal{D}u = \{ \check{r} \mid r \in \check{\mathbb{Q}} \}$ and $[\![u \subset \check{\mathbb{Q}}\,]\!] = 1$, and the power set of $\check{\mathbb{Q}}$ is denoted by $\mathcal{P}(\check{\mathbb{Q}})$, i.e.,

$$\mathcal{P}(\check{\mathbb{Q}}) = \{ x \mid x \subset \check{\mathbb{Q}} \}.$$

Definition 5.1. (in $V^{P(H)}$). $u \in \mathcal{P}(\check{\mathbb{Q}})$ is said to be a *real number*, if

$(D1)$ $\exists r \in \check{\mathbb{Q}}(r \in u) \wedge \exists r \in \check{\mathbb{Q}}(r \in u)^{\perp}$

$(D2)$ $\forall r \in \check{\mathbb{Q}} \big((r \in u) \longleftrightarrow \exists s \in \check{\mathbb{Q}}((s < r) \wedge (s \in u))\big)$

Lemma 5.2. *If $[\![u$ is a real number$]\!] = p \neq 0$, then there exists a complete Boolean subalgebra B of $P(H)$ such that $\{p\} \cup \{[\![\check{r} \in u]\!] | r \in \mathbb{Q}\} \subset B$.*

Proof: If $s, t \in \mathbb{Q}$ and $s \leqslant t$, then

$$[\![\forall r \in \check{\mathbb{Q}}(r \in u)]\!] \leqslant [\![\check{s} \in u]\!] \leqslant [\![\check{t} \in u]\!] \leqslant [\![\exists r \in \check{\mathbb{Q}}(r \in u)]\!]$$

by (D2). It follows that

$$M = \big\{[\![\exists r \in \check{\mathbb{Q}}(r \in u)]\!], [\![\forall r \in \check{\mathbb{Q}}(r \in u)]\!]\big\} \cup \{[\![\check{r} \in u]\!] \mid r \in \mathbb{Q}\}$$

is linearly ordered. Let B be the subalgebra of $P(H)$ generated by M. B is a complete Boolean subalgebra of $P(H)$ and $p \in B$. □

Lemma 5.3. *For $u \in V^{P(H)}$, if $p = [\![u$ is a real number$]\!]$ in $V^{P(H)}$ then there exists $v \in V^{P(H)}$ such that $\mathcal{D}v = \{ \check{r} \mid r \in \mathbb{Q} \}$ and*

$$p \leq [\![u \underset{q}{=} v]\!] \quad and \quad [\![v \textit{ is a real number}]\!] = 1.$$

Proof: Let

$$\mathcal{D}v = \{ \check{r} \mid r \in \mathbb{Q} \}, \quad v(\check{r}) = \begin{cases} [\![\check{r} \in u]\!] \wedge p, & r \leqslant 0, \\ ([\![\check{r} \in u]\!] \wedge p) \vee p^{\perp}, & 0 < r. \end{cases}$$

By Lemma 5.2, $\{p\} \cup \{[\![\check{r} \in u]\!] \mid r \in \mathbb{Q}\}$ is compatible. It follows that

$$p \leqslant [\![u \underset{q}{=} v]\!] \quad \text{and} \quad [\![v \text{ is a real number}]\!] = 1.$$

□

Definition 5.2. $u \in V^{P(H)}$ is called a *quantum real number* if

$$\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}, \quad \text{and} \quad [\![u \textit{ is a real number}]\!] = 1.$$

In $V^{P(H)}$, the set of all real numbers is denoted by $\mathfrak{R}$, i.e.,

$$\mathcal{D}\mathfrak{R} = \{\, u \in V^{P(H)} \mid u \textit{ is a quatum real number}\,\}, \quad \mathfrak{R}(u) = 1$$

Lemma 5.4. *If u, v are quantum real numbers, then*

$$[\![u \underset{q}{=} v]\!] = 1 \iff [\![u = v]\!] = 1.$$

Proof: It follows from the fact that $(a \to_q b) = 1 \iff (a \to b) = 1$ for $a, b \in P(H)$. □

5.2. Representation of Quantum Real Numbers

Theorem 5.1. (Takeuti (1978)). *If u is a quantum real number, then $E_u : \mathbb{R} \to P(H)$ defined by*

$$E_u(\lambda) = \bigwedge_{\lambda < r} [\![\check{r} \in u]\!]$$

is a resolution of the identity. That is, E_u satisfies

$$\bigwedge_{\lambda \in \mathbb{R}} E_u(\lambda) = 0, \quad \bigvee_{\lambda \in \mathbb{R}} E_u(\lambda) = 1, \quad E_u(\lambda) = \bigwedge_{\lambda < \mu} E_u(\mu).$$

Conversely, if $E : \mathbb{R} \to P(H)$ is a resolution of identity and $u \in V^{P(H)}$ is defined by

$$\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\} \quad \text{and } u(\check{r}) = \bigvee_{s < r} E(s)$$

then u is a quantum real number and $E_u = E$.

Proof: Let $[\![u$ is a real number less$]\!] = 1$. Then $r \leqslant s$ implies $[\![\check{r} \in u]\!] \leqslant [\![\check{s} \in u]\!]$. Hence, $E_u(r) = \bigwedge_{r<s} [\![\check{s} \in u]\!] \geqslant [\![\check{r} \in u]\!]$. It follows that

$$\bigvee_{\lambda \in \mathbb{R}} E_u(\lambda) \geqslant \bigvee_{r \in \mathbb{Q}} [\![\check{r} \in u]\!] = 1, \quad \bigwedge_{\lambda \in \mathbb{R}} E_u(\lambda) \leqslant \bigwedge_{r \in \mathbb{Q}} [\![\check{r} \in u]\!] = 0,$$

$$\text{and} \quad \bigwedge_{\lambda < \mu} E_u(\mu) = \bigwedge_{\lambda < \mu} \bigwedge_{\mu < r} [\![\check{r} \in u]\!] = \bigwedge_{\lambda < r} [\![\check{r} \in u]\!] = E_u(\lambda).$$

That is, E_u is a resolution of identity. Conversely, if E is a resolution of identity, let

$$\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\} \quad \text{and} \quad u(\check{r}) = \bigvee_{s<r} E(s) \quad \text{for} \quad r \in \mathbb{Q}.$$

It is obvious that u is a quantum real number. If $\lambda < r < \mu$, then

$$E(\lambda) \leqslant [\![\check{r} \in u]\!] \leqslant E_u(\mu) \;\therefore\; E(\lambda) \leqslant E_u(\mu),$$
$$E_u(\lambda) \leqslant [\![\check{r} \in u]\!] = \bigvee_{s<r} E(s) \leqslant E(\mu) \;\therefore\; E_\mu(\lambda) \leqslant E(\mu).$$

It follows that u is a quantum real number and $E_u = E$. □

Thus, quantum real numbers represent self-adjoint operators on H. The quantum real number representing self-adjoint operator A will be denoted by $\hat{A}$.

$$[\![\hat{A} = u]\!] = 1 \iff A = A_u.$$

5.3. Operations $+$ and $\cdot$ on $\mathfrak{R}$

Definition 5.3. Quantum real numbers u, v are said to be *compatible* if $\check{r} \in u$ and $\check{s} \in v$ are compatible for all $r, s \in \mathbb{Q}$, i.e.,

$$\forall r, s \in \hat{\mathbb{Q}}((\check{r} \in u) \downharpoonright (\check{s} \in v)).$$

If u, v are compatible quantum real numbers, then $u + v$ is defined by

$$u + v \overset{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \exists r_1, r_2 \in \check{\mathbb{Q}}((r = r_1 + r_2) \wedge (r_1 \in u) \wedge (r_2 \in v))\}.$$

For quantum real numbers u, v,

$$u \leqslant v \overset{\text{def}}{\iff} \forall r(r \in v \to r \in u).$$

u is said to be *positive* if $\check{0} \leqslant u$. If u, v are compatible positive quantum reals, then $u \cdot v$ is defined by

$$u \cdot v = \{r \in \check{\mathbb{Q}} \mid \exists r_1, r_2 \in \check{\mathbb{Q}}((r = r_1 \cdot r_2) \wedge (r_1 \in u) \wedge (r_2 \in v))\}.$$

The definition of $\cdot$ can be extended to the case that $\{u, v\}$ is any compatible set of qunantum real numbers and also quantum complex numbers.

Theorem 5.2. (Takeuti (1978)). *If u, v are compatible quantum real numbers, corresponding to self-adjoint operators A_u, A_v, respectively,*

$$A_u = \int \lambda d E_u(\lambda) \qquad A_v = \int \lambda d E_v(\lambda),$$

then

$$A_{u+v} = \int \lambda dE_{u+v}(\lambda) = \int \lambda dE_u(\lambda) + \int \lambda dE_v(\lambda) = A_u + A_v$$

$$A_{u\cdot v} = \int \lambda dE_{u\cdot v}(\lambda) = \int \lambda dE_u(\lambda) \cdot \int \lambda dE_v(\lambda) = A_u \cdot A_v$$

A complex number is defined in $V^{P(H)}$ as a pair $\langle u, v\rangle$ of compatible real numbers u, v. $\langle u, v\rangle$ is denoted by $u + iv$, where $i^2 = -\check{1}$. If u, v are compatible quantum real numbers, then $u + iv$ is said to be a *quantum complex number*. A quantum complex number represents a normal operator on H. We denote the set of quantum complex numbers by $\mathfrak{C}$ in $V^{P(H)}$.

6. PROPOSITIONS, SYMMETRIES, AND STATES IN $V^{P(H)}$

Quantum propositions are represented as projections which are self-adjoint operators acting on a Hilbert space H, hence represented as quantum real numbers. Symmetries are represented as unitary operators, and states of the quantum system are represented as unit vectors of H. The probability of obtaining the answer "yes" by carrying out a measurement corresponding to proposition p in initial state represented by a is $\|pa\|^2$. If $\|pa\|^2 \neq 0$, then the state immediately after the experiment is represented by $\frac{pa}{\|pa\|}$.

6.1. Representation of Quantum Propositions in $V^{P(H)}$

A projection $p \in P(H)$ is a self-adjoint operator,

$$p = \int \lambda dE(\lambda) = 0 \cdot E(0) + 1 \cdot (E(1) - E(0)) = E(0)^{\perp}.$$

Thus, the quantum real number $\hat{p}$ corresponding to proposition p is given by

$$\hat{p}(\check{r}) = \bigvee_{s<r} E(s) = \begin{cases} 0, & r \leqslant 0, \\ E(0) = p^{\perp}, & 0 < r \leqslant 1, \\ 1, & 1 < r, \end{cases}$$

Theorem 6.1. *If $p \in P(H)$, then $\hat{p}$ is a quantum real number such that*

$$[\![\hat{p} \underset{q}{=} \check{1}]\!] = [\![\hat{p} \underset{q}{=} \check{0}]\!]^{\perp} = p.$$

Conversely, if a positive quantum real u in $V^{P(H)}$ satisfies

$$[\![u \underset{q}{=} \check{1}]\!] = [\![u \underset{q}{=} \check{0}]\!]^{\perp} = p,$$

then $[\![u = \hat{p}]\!] = 1$.

Proof: Let $p \in P(H)$. It is obvious that $\hat{p}$ is a positive quantum real number and

$$(\hat{p}(\check{r}) \rightarrow_q [\![\check{r} \in \check{1}]\!]) \wedge (\check{1}(\check{r}) \rightarrow_q [\![\check{r} \in \hat{p}]\!]) = \begin{cases} 1, & r \leqslant 0, \\ p, & 0 < r \leqslant 1, \\ 1, & 1 < r, \end{cases}$$

$$(\hat{p}(\check{r}) \rightarrow_q [\![\check{r} \in \check{0}]\!]) \wedge (\check{0}(\check{r}) \rightarrow_q [\![\check{r} \in \hat{p}]\!]) = \begin{cases} 1, & r \leqslant 0, \\ p^{\perp}, & 0 < r \leqslant 1, \\ 1, & 1 < r. \end{cases}$$

Therefore, $[\![\hat{p} \underset{q}{=} \check{1}]\!] = [\![\hat{p} \underset{q}{=} \check{0}]\!]^{\perp} = p$.

Conversely, assume that u is a positive quantum real number such that

$$[\![u \underset{q}{=} \check{1}]\!] = [\![u \underset{q}{=} \check{0}]\!]^{\perp} = p.$$

Since $[\![u \subset \check{\mathbb{Q}}]\!] = 1$, we may assume that $\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}$. It suffices to show that

$$u(\check{r}) = \begin{cases} 0, & r \leqslant 0, \\ p^{\perp}, & 0 < r \leqslant 1, \\ 1, & 1 < r. \end{cases}$$

1. If $r \leqslant 0$, then

   $$p \leqslant (u(\check{r}) \rightarrow_q [\![\check{r} \in \check{1}]\!]) = u(\check{r})^{\perp} \quad \text{and} \quad p^{\perp} \leqslant (u(\check{r}) \rightarrow_q [\![\check{r} \in \check{0}]\!]) = u(\check{r})^{\perp}.$$

   It follows that $u(\check{r}) \leqslant (p \wedge p^{\perp}) = 0$.
2. If $1 < r$, then

   $$p \leqslant (\check{1}(\check{r})) \rightarrow_q [\![\check{r} \in u]\!]) = u(\check{r}) \quad \text{and} \quad p^{\perp} \leqslant (\check{0}(\check{r}) \rightarrow_q [\![\check{r} \in u]\!]) = u(\check{r}).$$

   Hence, $1 = (p \vee p^{\perp}) \leqslant u(\check{r})$.
3. If $0 < r \leqslant 1$, then

   $$p \leqslant (u(\check{r})) \rightarrow_q [\![\check{r} \in \check{1}]\!]) = u(\check{r})^{\perp} \quad \text{and} \quad p^{\perp} \leqslant (\check{0}(\check{r}) \rightarrow_q [\![\check{r} \in u]\!]) = u(\check{r}).$$

   Hence, $p^{\perp} \geqslant u(\check{r})$ and $p^{\perp} \leqslant u(\check{r})$. It follows that $u(\check{r}) = p^{\perp}$.

□

Thus, projection $p \in P(H)$ is represented as a quantum real number u such that

$$[\![u \underset{q}{=} \check{1}]\!] = [\![u \underset{q}{=} \check{0}]\!]^{\perp} = p.$$

Definition 6.1. (in $V^{P(H)}$). A quantum real number u such that

$$u \underset{q}{=} \check{1} \leftrightarrow (u \underset{q}{=} \check{0})^{\perp}$$

is called a *proposition*, and the set of propositions will be denoted by $\widehat{P(H)}$.

$$\widehat{P(H)} \stackrel{\text{def}}{=} \{\, u \in \mathfrak{R} \mid (u \underset{q}{=} \check{1}) \leftrightarrow (u \underset{q}{=} \check{0})^{\perp} \}.$$

That is,

$$\mathcal{D}(\widehat{P(H)}) = \{\, \hat{p} \mid p \in P(H) \}, \quad (\widehat{P(H)})(\hat{p}) = 1 \quad \text{for } p \in P(H)$$

Lemma 6.1.

$$[\![\widehat{P(H)} \underset{q}{=} \{\check{0}, \check{1}\}]\!] = 1$$

Proof: If $p \in P(H)$, then

$$\widehat{P(H)}(\hat{p}) = 1 = [\![\hat{p} \underset{q}{=} \check{1}]\!] \vee [\![\hat{p} \underset{q}{=} \check{1}]\!]^{\perp} = [\![\hat{p} \underset{q}{=} \check{1}]\!] \vee [\![\hat{p} \underset{q}{=} \check{0}]\!] \leqslant [\![\hat{p} \underset{q}{\in} \{\check{0}, \check{1}\}]\!].$$

Since $\check{0} = \hat{0} \in \widehat{P(H)}$ and $\check{1} \in \widehat{P(H)}$,

$$[\![\hat{p} \underset{q}{\in} \{\check{0}, \check{1}\}]\!] \leqslant [\![\hat{p} \underset{q}{=} \check{0} \vee \hat{p} \underset{q}{=} \check{1}]\!] \leqslant [\![\hat{p} \in \widehat{P(H)}]\!] = 1.$$

$\square$

For $p, q \in P(H)$, $\hat{p} \leqslant \hat{q} \iff \hat{q} \subset \hat{p}$. Hence,

Lemma 6.2. *For* $p, q \in P(H)$, $\quad p \leqslant q \iff [\![\hat{p} \leqslant \hat{q}]\!] = 1.$

Proof: Let $r \in \mathbb{Q}$. If $r \leqslant 0$ or $1 < r$, then $[\![\check{r} \in \hat{p}]\!] = [\![\check{r} \in \hat{q}]\!]$. If $0 < r \leqslant 1$, then

$$p \leqslant q \iff [\![\check{r} \in \hat{q}]\!] = q^{\perp} \leqslant p^{\perp} = [\![\check{r} \in \hat{p}]\!].$$

It follows that

$$p \leqslant q \iff \forall r \in \mathbb{Q}([\![\check{r} \in \hat{q} \to \check{r} \in \hat{p}]\!] = 1) \iff [\![\hat{p} \leqslant \hat{q}]\!] = 1.$$

$\square$

Lemma 6.3. *For* $A \subset P(H)$, $[\![\bigvee_{p \in A} \hat{p} = \widehat{\bigvee A}]\!] = 1.$

Proof: $\bigvee_{p \in A} \hat{p} \leqslant \widehat{\bigvee A}$ is obvious. If $\bigvee_{p \in A} \hat{p} \leqslant u \in \widehat{P(H)}$, then $u = \hat{q}$ for some $q \in P(H)$, and $\bigvee A \leqslant q$. Hence, $\widehat{\bigvee A} \leqslant \bigvee_{p \in A} \hat{p}$. $\square$

Lemma 6.4. *For* $p, q \in P(H)$,

$$[\![\hat{p} \underset{q}{=} \check{1} \leftrightarrow \hat{q} \underset{q}{=} \check{0}]\!] = 1 \iff p = q^{\perp}.$$

Proof: It follows from the fact that $p = [\![\hat{p} =_q \check{1}]\!]$ and $q^\perp = [\![\hat{q} =_q \check{1}]\!]^\perp = [\![\hat{q} =_q \check{0}]\!]$. $\square$

Definition 6.2. For $p \in P(H)$,

$$\hat{p}^\perp \stackrel{\text{def}}{=} \widehat{p^\perp}$$

Definition 6.3. (in $V^{P(H)}$). $u, v \in \widehat{P(H)}$ are said to be *orthogonal* if

$$u =_q \check{1} \to v =_q \check{0}.$$

$u \in \widehat{P(H)}$ is called an *atom* if

$$\forall b \in \widehat{P(H)} (\check{0} < b \leqslant u \to b = u).$$

The set of atoms is denoted by **Atom**.

$$\textbf{Atom} \stackrel{\text{def}}{=} \{u \in \widehat{P(H)} \mid \forall b \in \widehat{P(H)} (\check{0} < b \leqslant u \to b = u)\}.$$

Axiom 12.

A1: $\forall b \in \widehat{P(H)} \exists p \in \textbf{Atom}\ (p \leqslant b)$.

A2: $\forall b, q \in \widehat{P(H)} \forall p \in \textbf{Atom}\ [(p \wedge b = 0) \wedge (b \leqslant q < p \vee b) \to (b = q)]$

(A1) and (A2) are valid in $V^{P(H)}$.

6.2. Subuniverse V^{B_γ} of $V^{P(H)}$

From now on we fix a basis $\{e_i\}_{i \in I}$ of H and denote the projection on the 1-dimensional subspace containing e_i by p_i, for each $i \in I$. Let $\mathfrak{U} = \{U_\gamma\}_{\gamma \in \Gamma}$ be the set of all unitary opertors acting on H.

For each unitary operator U_γ, $\{U_\gamma p_i U_\gamma^*\}_{i \in I}$ generates a complete Boolean subalgebra of $P(H)$, which we denote by B_γ. The identity mapping $B_\gamma \to P(H)$ is extended to the identity mapping $V^{B_\gamma} \to V^{P(H)}$. That is, B_γ-valued universe V^{B_γ} is a subuniverse of $V^{P(H)}$, and

$$[\![u = v]\!]_{B_\gamma} = [\![u = v]\!] \quad \text{and} \quad [\![u \in v]\!]_{B_\gamma} = [\![u \in v]\!] \quad (u, v \in V^{B_\gamma}).$$

Definition 6.4. For a unitary operator $U \in \mathfrak{U}$, The mapping

$$\sigma_U : p \mapsto UpU^*$$

is an automorphism of $P(H)$, and is extended to the automorphism of universe

$$\sigma_U : V^{P(H)} \to V^{P(H)}.$$

By Theorem 3.2, we have

Theorem 6.2. *For each unitary operator $U \in \mathfrak{U}$,*

1. $\sigma_U[\![u = v]\!] = [\![\sigma_U u = \sigma_U v]\!]$ *and* $\sigma_U[\![u \in v]\!] = [\![\sigma_U u \in \sigma_U v]\!]$ *for* $u, v \in V^{P(H)}$.
2. *If* $\varphi(u_1, \ldots, u_n)$ *is a bounded formula with constants* $u_1, \ldots, u_n$, *then* $\sigma_U[\![\varphi(u_1, \ldots, u_n)]\!] = [\![\varphi(\sigma_U(u_1), \ldots, \sigma_U(u_n))]\!]$.
3. $\sigma_U[\![u \underset{q}{=} v]\!] = [\![\sigma_U u \underset{q}{=} \sigma_U v]\!]$.

Corollary 6.1. *If $p \in P(H)$ and $U \in \mathfrak{U}$, then projection p is represented by quantum real number $\hat{p}$, and $\sigma_U(\hat{p})$ is the quantum real number $\widehat{UpU^*}$ corresponding to $\sigma_U(p) = UpU^*$.*

Proof: $[\![\hat{p} \underset{q}{=} \check{1}]\!] = p$ implies $[\![\sigma_U(\hat{p}) \underset{q}{=} \check{1}]\!] = \sigma_U(p) = UpU^*$. □

A quantum complex number $u + iv$ represents a normal operator. The compatible quantum real numbers u and v correspond to self-adjoint operators $A_u = \int \lambda dE_u(\lambda)$ and $A_v = \int \lambda dE_v(\lambda)$, and there exists a unitary operator U_γ on H such that each element of

$$\{E_u(\lambda) \mid \lambda \in \mathbb{R}\} \cup \{E_v(\lambda) \mid \lambda \in \mathbb{R}\}$$

is spanned by a subset of $\{U_\gamma e_i\}_{i \in I}$ (cf. Halmos, 1951). That is, there exists a unitary operator $U_\gamma \in \mathfrak{U}$ such that $u + iv \in V^{B_\gamma}$.

Lemma 6.5. (Takeuti (1978)). *Let u, v be quantum real numbers in $V^{P(H)}$. If there is a unitary opertor $U_\gamma \in \mathfrak{U}$ such that $u, v \in V^{B_\gamma}$, then for $p \in B_\gamma$*

$$p \leqslant [\![u \underset{q}{=} v]\!] \iff p \cdot A_u = p \cdot A_v,$$

where A_u and A_v are self-adjoint operators on H corresponding to u, v, respectively.

6.3. Representation of Symmetries in $V^{P(H)}$

A symmetry in the physical system is represented as a unitary operator of H, which is a normal operator. Hence, each symmetry is represented as a quantum complex number. The quantum complex number representing unitary operator U will be denoted by $\hat{U}$. $\mathfrak{U} = \{U\}_{\gamma \in \Gamma}$ denotes the set of all unitary operators acting on H. The set of quantum complex numbers representing $\mathfrak{U}$ in $V^{P(H)}$ will be denoted by $\hat{\mathfrak{U}}$, i.e.,

$$\mathcal{D}\hat{\mathfrak{U}} = \{\hat{U} | U \in \mathfrak{U}\}, \qquad \hat{\mathfrak{U}}(\hat{U}) = 1 \quad (U \in \mathfrak{U})$$

Each unitary operator $U \in \mathfrak{U}$ induces an automorphism of $P(H)$,

$$\sigma_U : p \mapsto UpU^*,$$

which preserves $\bigvee$ and $^\perp$. σ_U is extended to an automorphism $\sigma_U : V^{P(H)} \to V^{P(H)}$ of universe.

Generally, "f is a mapping of A to B" in $V^{P(H)}$, in symbols $f : A \to B$, is defined as

$$f : A \to B \overset{\text{def}}{\iff} f \subset A \times B \wedge \forall a \in A \exists b \in B(\langle a, b\rangle \in f) \wedge$$
$$\forall a \in A \forall b, c \in B(\langle a, b\rangle \in f \wedge \langle a, c\rangle \in f \to b = c)$$

$A \in V^{P(H)}$ is said to be *global* if $A(x) = 1$ or $A(x) = 0$ for each $x \in \mathcal{D}A$. If $A \in V^{P(H)}$ is global, we denote the set $\{x \in \mathcal{D}A | A(x) = 1\}$ by $\tilde{A}$.

Lemma 6.6. *If $A, B \in V^{P(H)}$ are global, and $\tilde{\varphi}$ is a mapping $\tilde{A} \to \tilde{B}$, then $\varphi \in V^{P(H)}$ defined by*

$$\mathcal{D}\varphi = \{\langle x, \tilde{\varphi}(x)\rangle | x \in \tilde{A}\}, \quad \varphi(\langle x, \tilde{\varphi}(x)\rangle) = 1$$

satisfies $[\![\varphi : A \to B]\!] = 1$. We denote $\langle x, y\rangle \in \varphi$ by $\varphi(x) = y$.

Since $\widehat{P(H)}$ is global, $\sigma_U : \hat{p} \mapsto \widehat{UpU^*}$ is the mapping in $V^{P(H)}$ representing a symmetry associated with unitary operator U.

6.4. Representation of States in $V^{P(H)}$

Definition 6.5. A set $\{\widehat{q_i}\}_{i \in I}$ of mutually orthogonal atoms of $\widehat{P(H)}$ is said to be a *complete system of atoms* if $\bigvee_{i \in I} \hat{q}_i = \hat{I}$.

For each unitary operator $U \in \mathfrak{U}$, $\{\sigma_U(\hat{p}_i)\}_{i \in I}$ is a complete system of atoms. If c_i is a complex number for each $i \in I$, and $U_\gamma \in \mathfrak{U}$, then $\check{c}_i$ is a quantum complex number in V^{B_γ}, and $\{\sigma_{U_\gamma}(\hat{p}_i)\}_{i \in I}$ is a set of mutually compatible quantum real numbers in V^{B_γ}, hence, $\sum_{i \in I} \check{c}_i \cdot \sigma_{U_\gamma}(\hat{p}_i)$ is a quantum complex number in V^{B_γ}.

Definition 6.6. If $\{c_i\}_{i \in I} \subset \mathbb{C}$, $\sum_{i \in I} |c_i|^2 < \infty$ and $U \in \mathfrak{U}$, then $u = \sum_{i \in I} \check{c}_i \cdot \sigma_U(\hat{p}_i)$ represents the normal operator $\sum_{i \in I} c_i \cdot U p_i U^*$ (Corollary 6.1). We denote $(\sqrt{\sum_{i \in I} |\check{c}_i|^2})$ by $\|u\|$.

Theorem 6.3. *If $\{b_i\}_{i \in I}, \{c_i\}_{i \in I} \subset \mathbb{C}$ and if $U_\gamma \in \mathfrak{U}$, then quantum complex numbers $\sum_{i \in I} \check{b}_i \cdot \sigma_{U_\gamma}(\hat{p}_i)$ and $\sum_{i \in I} \check{c}_i \cdot \sigma_{U_\gamma}(\hat{p}_i)$ are elements of V^{B_γ}, and*

$$\left[\!\!\left[\sum_{i \in I} \check{b}_i \cdot \sigma_{U_\gamma}(\hat{p}_i) = \sum_{i \in I} \check{c}_i \cdot \sigma_{U_\gamma}(\hat{p}_i) \right]\!\!\right] = 1 \Longrightarrow b_i = c_i \quad \textit{for each } i \in I.$$

Proof: Let $U = U_\gamma$. $[\![\sum_{i\in I} \check{b}_i \cdot \sigma_U(\hat{p}_i) = \sum_{i\in I} \check{c}_i \cdot \sigma_U(\hat{p}_i)]\!] = 1$ implies

$$\sum_{i\in I} b_i \cdot \sigma_U(p_i) = \sum_{i\in I} c_i \cdot \sigma_U(p_i).$$

Then, by Lemma 6.5, $\sigma_U(p_i) = Up_iU^* \leqslant [\![\check{b}_i \underset{q}{=} \check{c}_i]\!]$. It follows that $b_i = c_i$ for each $c \in I$. □

States of the physical system are represented by unit vectors of Hilbert space H. Each vector $x \in H$ has expressions of the form $\sum_{i\in I}(x, Ue_i)Ue_i$ for each unitary operator U. For $x \in H$ and $\gamma \in \Gamma$, let $\hat{x}_\gamma$ be the quantum complex number $\sum_{i\in I}(x, U_\gamma e_i)\check{} \cdot \sigma_{U_\gamma}(\hat{p}_i)$, which is in V^{B_γ}.

$$\hat{x}_\gamma = \sum_{i\in I}(x, U_\gamma e_i)\check{} \cdot \sigma_{U_\gamma}(\hat{p}_i).$$

$x \in H$ will be represented as set $[x] = \{\hat{x}_\gamma\}_{\gamma\in\Gamma}$ of quantum complex numbers indexed by Γ. Indexed set $\{\hat{x}_\gamma\}_{\gamma\in\Gamma}$ means the mapping from check set Γ to the set $\mathfrak{C}$ of quantum complex numbers defiend by $\gamma \mapsto \hat{x}_\gamma$. Let

$$[x] = \{\hat{x}_\gamma\}_{\gamma\in\Gamma}, \quad \text{where } \hat{x}_\gamma = \sum_{i\in I}(x, U_\gamma e_i)\check{} \cdot \sigma_{U_\gamma}(\hat{p}_i).$$

$$H_\gamma = \{\hat{x}_\gamma | x \in H\} \quad (\gamma \in \Gamma).$$

Lemma 6.7. *If $x, y \in H$ and $\gamma \in \Gamma$, then*

$$[\![\hat{x}_\gamma = \hat{y}_\gamma]\!] = 1 \iff [x] = [y].$$

Proof: Let $U = U_\gamma$. By Theorem 6.3,

$$\begin{aligned}\left[\!\!\left[\sum_{i\in I}(x, Ue_i)\check{} \cdot \sigma U(\hat{p}_i) \right.\right. &= \left.\left.\sum_{i\in I}(y, Ue_i)\check{} \cdot \sigma U(\hat{p}_i)\right]\!\!\right] = 1 \\ &\Longrightarrow (x, Ue_i) = (y, Ue_i) \quad \forall i \in I \\ &\Longrightarrow x = y\end{aligned}$$

□

It is obvious that

Lemma 6.8. *For $x, y \in H$ and $c \in \mathbb{C}$,*

$$(\widehat{x+y})_\gamma = \hat{x}_\gamma + \hat{y}_\gamma, \qquad (\widehat{cx})_\gamma = \check{c} \cdot \hat{x}_\gamma, \qquad \|\hat{x}_\gamma\| = \|x\|\check{},$$

where $\|\hat{x}_\gamma\| = \|\sum_i (x, Ue_i)\check{} \cdot \sigma_{U_\gamma}(\hat{p}_i)\| = \left(\sqrt{\sum_i |(x, U_\gamma e_i)|^2}\right)\check{}.$

Definition 6.7.

$$[x] + [y] \stackrel{\text{def}}{=} [x + y], \quad \check{c} \cdot [x] \stackrel{\text{def}}{=} [cx], \quad \|[x]\| \stackrel{\text{def}}{=} \|x\|\check{}.$$

$\|[x]\|$ will be denoted simply by $\|x\|$.

Definition 6.8.

$$\hat{H} \stackrel{\text{def}}{=} \{[x] \mid x \in H\}.$$

We say $[x] \in \hat{H}$ is a state if $\|x\| = \check{1}$.

$$\textbf{State} \stackrel{\text{def}}{=} \{[x] \in \hat{H} \mid \|x\| = \check{1}\}$$

If x is a nonzero vector in H, then $\frac{[x]}{\|x\|}$ is a state.

Theorem 6.4. *For a projection $p \in P(H)$, there exists $\gamma \in \Gamma$ such that $p \downharpoonleft \{U_\gamma p_i U_\gamma^*\}_{i \in I}$. Then $\hat{P} \in V^{B_\gamma}$, and*

$$\hat{p} \cdot \hat{x}_\gamma = (\widehat{px})_\gamma.$$

Proof: Let $U = U_\gamma$. It is obvious that $p \downharpoonleft \{U p_i U^*\}_{i \in I}$ implies $p \in B_\gamma$ and $\hat{p} \in V^{B_\gamma}$. Hence, $\hat{p}$ and $\hat{x}_\gamma$ are compatible.

$$\begin{aligned} \hat{p} \cdot \hat{x}_\gamma &= \sum_{i \in I} (x, Ue_i)\check{} \cdot \hat{p} \cdot \sigma_U(\hat{p}_i) \\ &= \sum_{U p_i U^* \leqslant p} (x, Ue_i)\check{} \cdot \sigma_U(\hat{p}_i) = (\widehat{px})_\gamma. \end{aligned}$$

□

Definition 6.9. $\hat{p}$ defines the mapping $\hat{p} : \hat{H} \to \hat{H}$ by

$$\hat{p}[x] \stackrel{\text{def}}{=} [px]$$

Theorem 6.5. *Let $U_\gamma \in \mathfrak{U}$. For each $i \in I$,*

$$(\widehat{U_\gamma e_i})_\gamma = \sigma_{U_\gamma}(\hat{p}_i), \quad \textit{and} \quad [U_\gamma e_i] = \sum_{j \in I} (U_\gamma e_i, e_j)\check{}[e_j].$$

Proof:

$$(\widehat{U_\gamma e_i})_\gamma = \sum_{j \in I} (U_\gamma e_i, U_\gamma e_j) \sigma_{U_\gamma}(\hat{p}_j) = \sigma_{U_\gamma}(\hat{p}_i).$$

$$[U_\gamma e_i]_\gamma = \left[\sum_{j\in I}(U_\gamma e_i, e_j)e_j\right] = \sum_{j\in I}(U_\gamma e_i, e_j)\breve{}[e_j].$$

□

Definition 6.10. We denote $[Ue_i]$ by $\hat{U}[e_i]$. $\hat{U} \in \hat{\mathfrak{U}}$ defines the mapping $\hat{U}: \hat{H} \to \hat{H}$ by

$$\hat{U}[x] \stackrel{\text{def}}{=} [Ux] = \sum_{j\in I}(x, e_j)\breve{}[Ue_j].$$

For the experiment of $p \in P(H)$ in state $[a]$,

$$[a] = [pa + p^\perp a] = [pa] + [p^\perp a].$$

If $pa = 0$ or $p^\perp a = 0$, then

$$[a] = \|pa\| \cdot [pa] + \|p^\perp a\| \cdot [p^\perp a].$$

If $pa \neq 0$ and $p^\perp a \neq 0$, then

$$[a] = \|pa\| \cdot [x] + \|p^\perp a\| \cdot [y],$$

where $[x] = \frac{[pa]}{\|pa\|}$ and $[y] = \frac{[p^\perp a]}{\|p^\perp a\|}$ are states, and $\|pa\|^2$ ($\|p^\perp a\|^2$) is the probability of obtaining "yes" ("no").

6.5. Applications

We consider the experiment which consists in placing a polarizer in the beam of photons. When the photons are despatched one by one, this experiment leads to a plain alternative: either the photon passes through, or it is absorbed. We shall define the proposition p_θ by specifying the orientation of the polarizer (the angle θ) and interpreting the passage of a photon as a "yes." Experiment shows that, to obtain a photon prepared in such a way that "p_θ is true," it is sufficient to consider the photons which have traversed a first polarizer oriented at this angle θ. Let a_θ be the state in which a photon has traversed a first polarizer oriented at angle θ, and let the second polarizer be oriented at angle θ'. Then the probability $\|p_{\theta'}a_\theta\|^2$ of "yes" depends only on the difference of the angles $\theta' - \theta$, and is known to be

$$\cos^2(\theta' - \theta).$$

The experiment is interpreted in $V^{P(H)}$ as follows.

The measurement of $p_{\theta'}$ in state a_θ induces a change of expression of the state.

$$[a_\theta] = [p_{\theta'}a_\theta] + [(p_{\theta'})^\perp a_\theta]$$

where $\| p_{\theta'} a_\theta \|^2 = \cos^2(\theta' - \theta)$ is the probability of "yes" and $\|(p_{\theta'})^\perp a_\theta\|^2 = \sin^2(\theta' - \theta)$ is the probability of "no."

Let $\{a_0, a_{\frac{\pi}{2}}\}$ be the fixed basis of H. If $\theta = 0$ and $\theta' = \frac{\pi}{2}$, then

$$[a_0] = \left[p_{\frac{\pi}{2}} a_0\right] + \left[p_{\frac{\pi}{2}}^\perp a_0\right] = \left(\cos \frac{\pi}{2}\right)\check{}\left[a_{\frac{\pi}{2}}\right] + (\cos\theta)\check{}[a_0].$$

Since $(\hat{a}_0)_I = \check{0} \cdot (\hat{a}_{\frac{\pi}{2}})_I + \check{1} \cdot (\hat{a}_0)_I = \check{0} \cdot \hat{p}_{\frac{\pi}{2}} + \check{1} \cdot \hat{p}_\theta$, where I is the identity operator, we have

$$p_{\frac{\pi}{2}} \leqslant [\![(\hat{a}_0)_I \underset{q}{=} \check{0}]\!], \quad \text{and} \quad p_{\frac{\pi}{2}} \leqslant [\![(\hat{a}_0)_I \underset{q}{=} \check{1}]\!].$$

This means that right after the traversing in the state $[a_0]$, the answer for $p_{\frac{\pi}{2}}$ is "yes" with probability 0, that is, the answer is "no," and the answer for $(p_{\frac{\pi}{2}})^\perp = p_0$ is "yes" with probability 1.

If $\theta = \frac{\pi}{4}$ and $\theta' = \frac{\pi}{2}$, then $\|p'_\theta a_\theta\|^2 = \frac{1}{2}$, $\|(p'_\theta)^\perp a_\theta\|^2 = \frac{1}{2}$, and

$$\left[a_{\frac{\pi}{4}}\right] = \left(\frac{1}{\sqrt{2}}\right)\check{}\left[a_{\frac{\pi}{2}}\right] + \left(\frac{1}{\sqrt{2}}\right)\check{}[a_0],$$

where $[a_{\frac{\pi}{2}}] = [\sqrt{2} p_{\frac{\pi}{2}} a_{\frac{\pi}{4}}]$ and $[a_0] = [\sqrt{2}(p_{\frac{\pi}{2}})^\perp a_{\frac{\pi}{4}}]$ are states. Thus,

$$p_{\frac{\pi}{2}} \leqslant \left[\!\!\left[(\hat{a}_{\frac{\pi}{4}})_I \underset{q}{=} \left(\frac{1}{\sqrt{2}}\right)\check{}\right]\!\!\right], \quad \text{and} \quad \left(p_{\frac{\pi}{2}}\right)^\perp \leqslant \left[\!\!\left[\left(\hat{a}_{\frac{\pi}{4}}\right)_I \underset{q}{=} \left(\frac{1}{\sqrt{2}}\right)\check{}\right]\!\!\right].$$

This means that right after the traversing in the state $[a_{\frac{\pi}{4}}]$, the answer for $p_{\frac{\pi}{2}}$ is "yes" with probability $\frac{1}{2}$, and "no" with probability $\frac{1}{2}$. The states right after measurement is $[a_{\frac{\pi}{2}}]$ or $[a_0]$, according to "yes" or "no."

ACKNOWLEDGMENTS

We are very grateful to Professor G. Takeuti for introducing us to the quantum set theory, and for his valuable advice. We also would like to express our thanks to our colleagues Professor H. Kodera and Professor H. Aoyama who have joined our seminar for useful discussion.

REFERENCES

Birkhoff, G. and von Neumann, J. (1963). The logic of quantum mechanics. *Annals of Mathematics* **37,** 823.

Halmos, P. R. (1951). *Introduction to Hilbert Space*, Chelsea, New York.

Piron, C. P. (1976). *Foundations of Quantum Physics*, W. A. Benjamin, New York.

Takano, M. (2002). Strong completeness of lattice-valued logic. *Archive for Mathematical Logic* **41,** 497–505.

Takeuti, G. (1978). *Two Applications of Logic to Mathematics*, Iwanami, Tokyo.

Takeuti, G. (1981). In *Qunatum Set Theory*, *Current Issues in Quantum Logic*, E. Beltrametti and B. C. van Frassen, eds., Plenum, New York, pp. 303–322.

Titani, S. (1999). Lattice-valued set theory. *Archive for Mathematical Logic* **38**(6), 395–421.

von Neumann, J. (1955). *Mathematical Foundation of Quantum Mechanics*, Princetion University Press, Princeson, NJ.